

Gravity as a four dimensional algebraic quantum field theory

Gábor Etesi

*Department of Geometry, Mathematical Institute, Faculty of Science,
Budapest University of Technology and Economics,
Egry J. u. 1, H ép., H-1111 Budapest, Hungary **

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Abstract

In this note a unitary representation of the orientation-preserving diffeomorphism group of an oriented 4-manifold is exhibited. More precisely the resulting representation space is a Banach space equipped with a non-degenerate indefinite Hermitian scalar product and as a vector space admits a family of corresponding direct sum decompositions into orthogonal pairs of maximal definite Hilbert subspaces. It is observed that the associated “net of algebras of local quantum observables” on this representation space in the sense of algebraic quantum field theory contains curvature tensors. Classical vacuum gravitational fields i.e., Einstein manifolds correspond to quantum observables obeying at least one of the above decompositions of the space. General local quantum observables of this theory are also investigated.

In this way classical general relativity exactly in 4 dimensions naturally embeds into an algebraic quantum field theory possessing a diffeomorphism group symmetry and this theory is constructed out of the structures provided by an oriented 4-manifold only.

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1 Introduction

The outstanding problem of modern theoretical physics is how to unify the obviously successful and mathematically consistent *theory of general relativity* with the obviously successful but yet mathematically problematic *relativistic quantum field theory*. It has been generally believed that these two fundamental pillars of modern theoretical physics contradict each other not only in the mathematical tools they use but even at a deep foundational level [6]: classical concepts of general relativity such as the space-time event, the light cone or the event horizon of a black hole are too “sharp” objects from a quantum theoretic viewpoint meanwhile relativistic quantum field theory is not background independent from the aspect of general relativity. Of course we do not attempt here to survey the vast physical and even mathematical and philosophical literature created by the unification problem; we just mention

*e-mail: etesi@math.bme.hu

that nowadays the two leading candidates expected to be capable for a sort of unification are string theory and loop quantum gravity. But surely there is still a long way ahead.

Nevertheless we have the conviction that one day the language of classical general relativity will sound familiar to quantum theorists and vice versa i.e., conceptual bridges must exist connecting the two theories. In this note as a guiding principle the “diffeomorphism invariance of general relativity” has been taken seriously and an effort has been made to embed classical general relativity into a quantum framework. This quantum framework is provided by *algebraic quantum field theory* formulated by Haag–Kastler and others during the past decades, cf. [5]. Recently the same framework appears to be suitable for quantum field theory on curved space-time [8] or even quantum gravity [2].

In more detail we will do something very simple here. Namely using structures given by an oriented smooth 4-manifold M only, our primary aim will be seeking unitary representations of the corresponding orientation-preserving diffeomorphism group $\text{Diff}^+(M)$. It is surprising that there is a unique such representation on the space of sections of $\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C}$ via pullback. However the natural scalar product on this space—namely the one given by integration of the wedge product of two 2-forms—is indefinite hence cannot be used to complete the space of smooth 2-forms into a Hilbert space. Rather in struggling with the completion problem one comes up with a Banach space $\mathcal{V}(M)$ and a non-degenerate indefinite Hermitian scalar product $(\cdot, \cdot)_{L^2(M)}$ on it such that the bare space—i.e., not considered as an $\text{Diff}^+(M)$ -module—admits decompositions into maximal definite orthogonal Hilbert subspaces

$$\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$$

with respect to the scalar product.

Given this vector space carrying an indefinite unitary representation of the diffeomorphism group one would like to consider it as the state space of some quantum field theory over M possessing a diffeomorphism invariance. Therefore the next obvious task is to look for local quantum observables i.e., linear operators on this space. Although the constructed space $\mathcal{V}(M)$ is not a Hilbert space it can be completed into a Banach space in a unique way hence the easiest way is to consider the C^* -algebra of its bounded linear operators. More precisely given a relatively compact open subset $U \subseteq M$ we take the assignment $U \mapsto \mathcal{B}(U)$ consisting of bounded linear operators somehow associated to U so that the collection $\{U \mapsto \mathcal{B}(U)\}_{U \subseteq M}$ fit together into a “net of algebras” or a “net of local quantum observables” over M in the spirit of algebraic quantum field theory. So far our construction is merely representation theory spiced with quantum physics. However it comes as a surprise (at least to the author) that exactly in 4 dimensions among the local quantum observables one can discover curvature tensors! This is because of the well-known fact that the (complexified) curvature tensor R_g of a pseudo-Riemannian 4-manifold (M, g) can be regarded as a section of $\text{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})$ i.e., gives rise to a linear operator $R_M \in \mathcal{B}(M)$ acting on $\mathcal{V}(M)$. In this way classical vacuum general relativity naturally embeds into this algebraic quantum field theory. The appearance of the curvature tensor as a local quantum observable is reasonable from the physical viewpoint: in gravitational physics the metric tensor has no direct physical meaning only its curvature can cause *local* physical effects such as tidal forces. In fact at least in principle geometry locally can be reconstructed from its curvature [4].

Returning to the mathematical side, $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$ has an extra structure: a family of orthogonal splittings into positive and negative definite subspaces $\mathcal{H}^{\pm}(M)$ as we have already mentioned. Consequently it is natural to ask which curvature tensors obey at least one of these splittings i.e., have the form

$$R_M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}^+(M) \\ \mathcal{H}^-(M) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}^+(M) \\ \mathcal{H}^-(M) \end{pmatrix}$$

with respect to some direct sum decomposition of $\mathcal{V}(M)$? It turns out that these are exactly the curvatures of Einstein spaces i.e., solutions of the vacuum Einstein equations (with a cosmological constant).

Therefore we cannot resist the temptation to say that a general local quantum observable $Q_U \in \mathcal{B}(U)$ satisfies the *quantum vacuum Einstein equation* if it has the blockdiagonal form as above. Unlike the classical vacuum Einstein equation this quantum generalization is linear in accordance with the spirit of quantum theory and can be interpreted as a *superselection rule* for quantized pure gravity.

One may ask further how generic local quantum observables look like i.e., those which do not come from curvature tensors of Einstein metrics. We will demonstrate that they arise by a smearing procedure of classical curvature tensors hence the concept of a point of the space-time as a physical event becomes meaningless in the quantum context. Or conversely, given a solution (M, g) of the classical vacuum Einstein equation one may try to “quantize” it by smearing its curvature tensor into a generic bounded linear operator.

Finally we make a comment on the distinction between *Riemannian* and *Lorentzian* general relativity. Or equivalently the difference between *geometry* and *causality*. The causal future $J^+(p) \subset M$ of an event $p \in M$ in space-time is by definition the hull of all future-inextendible worldlines of particles departing from p and moving forward in time locally not faster than light. The causal past $J^-(p) \subset M$ is defined similarly. The collection of these causal futures and pasts furnish the space-time with a special topology, the causal structure. The Lorentzian metric is a mathematical fusion of the geometry captured by a Riemannian metric *and* the causal structure captured by a “causal topology”. But from this well-known operational description it is clear that beyond the gravitational field itself the construction of a causal structure refers to other elements of physical reality as well which are moreover classical: pointlike particles, electromagnetic waves, time, etc. These cannot be present in a vacuum space-time considered in the strict sense. Very strictly speaking even the interpretation of a space-time point as a “physical event” fails in an *empty* space-time. Henceforth we are convinced that causality cannot be a fundamental feature of a fully quantum description of pure gravity. As a consequence we will prefer to use Riemannian metrics in this note in dealing with the pure vacuum. From our standpoint causality is an emergent phenomenon created by the highly complex interaction of gravity and matter. Consequently in order to understand it first we should include matter in our model. We will not carry out this in this short note but observe that in light of Einstein’s equation the local quantum observable corresponding to gravity coupled to matter must be a non-self-adjoint operator which breaks the aforementioned superselection rule between $\mathcal{H}^+(M)$ and $\mathcal{H}^-(M)$; the mixing term is $8\pi T_0$ in the classical situation, the traceless part of the energy-momentum tensor of matter.

This note is organized as follows. In Sect. 2 we provide the mathematical details of the aforementioned program. After constructing a natural indefinite unitary representation of the orientation-preserving diffeomorphisms of an oriented 4-manifold on a direct sum Hilbert space, the broadest possible algebraic quantum field theory is assigned to this direct sum space. Then we explore the local quantum observables of this theory and identify the classical ones as classical Einstein spaces (see item **C** in Sect. 2) and the semiclassical ones as manifolds with local Einstein structures (see item **SC** in Sect. 2). Finally we formulate (see Lemma 2.4 here) how generic local quantum observables should result from classical ones by a “smearing procedure” based on the Schwartz kernel theorem. We summarize the properties of the resulting “quantum structure” in item **Q** in Sect. 2.

In Sect. 3 we proceed the other way round and begin to quantize classical Einstein manifolds by smearing their curvature tensors. Explicit calculations have been carried out for the flat \mathbb{R}^4 the round S^4 and the Riemannian Schwarzschild space.

Finally in Sect. 4 we conclude by asking ourselves how to generalize this model to include matter hence hopefully to recover the causal structure of space-time, too.

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2 Quantization of pure gravity

Let M be a connected orientable smooth 4-manifold, possibly non-closed (i.e., it can be non-compact and-or with non-empty boundary). Fix an orientation on M . Given only these data at our disposal it is already meaningful to talk about the group of its orientation-preserving diffeomorphisms $\text{Diff}^+(M)$. Our guiding principle simply will be a search for a unitary representation of $\text{Diff}^+(M)$. A bunch of representations arise in a geometric way as follows. Consider $T^{(r,s)}M \otimes_{\mathbb{R}} \mathbb{C}$, the bundle of complexified (r,s) -type tensors with the associated vector spaces $C_c^\infty(M; T^{(r,s)}M \otimes_{\mathbb{R}} \mathbb{C})$ of their compactly supported smooth complexified sections. Then the group $\text{Diff}^+(M)$ acts from the left via pushforward on $C_c^\infty(M; T^{(r,0)}M \otimes_{\mathbb{R}} \mathbb{C})$ for all $r \in \mathbb{N}$ while from the right via pullback on $C_c^\infty(M; T^{(0,s)}M \otimes_{\mathbb{R}} \mathbb{C})$ for all $s \in \mathbb{N}$. However these representations are typically not unitary because the underlying vector spaces do not carry extra structures in a natural way.

The only exception is the 2nd exterior power $\wedge^2 M \subset T^{(0,2)}M$ of the cotangent bundle with the corresponding space of sections $C_c^\infty(M; \wedge^2 M \otimes_{\mathbb{R}} \mathbb{C}) =: \Omega_c^2(M; \mathbb{C})$, the space of complexified smooth 2-forms with compact support. Indeed, this vector space has a natural non-degenerate Hermite scalar product $(\cdot, \cdot)_{L^2(M)} : \Omega_c^2(M; \mathbb{C}) \times \Omega_c^2(M; \mathbb{C}) \rightarrow \mathbb{C}$ given by integration on oriented smooth manifolds; more precisely for $\alpha, \beta \in \Omega_c^2(M; \mathbb{C})$ put

$$(\alpha, \beta)_{L^2(M)} := \int_M \bar{\alpha} \wedge \beta \quad (1)$$

(complex conjugate-linear in its first variable). Note however that this scalar product is *indefinite*: an unavoidable fact which plays a key role in our considerations ahead. Consequently this scalar product cannot be used to complete $\Omega_c^2(M; \mathbb{C})$ into a Hilbert space. Instead with respect to (1) there is a non-unique direct sum decomposition

$$\Omega_c^2(M; \mathbb{C}) = \Omega_c^+(M; \mathbb{C}) \oplus \Omega_c^-(M; \mathbb{C})$$

with the property that $\Omega_c^+(M; \mathbb{C}) \perp_{L^2(M)} \Omega_c^-(M; \mathbb{C})$ and $\Omega_c^\pm(M; \mathbb{C})$ are the maximal definite subspaces i.e., the signed restricted scalar products $\pm(\cdot, \cdot)_{L^2(M)}|_{\Omega_c^\pm(M; \mathbb{C})} : \Omega_c^\pm(M; \mathbb{C}) \times \Omega_c^\pm(M; \mathbb{C}) \rightarrow \mathbb{C}$ are both positive definite. Therefore these restricted scalar products can be used to complete $\Omega_c^\pm(M; \mathbb{C})$ into Hilbert spaces $\mathcal{H}^\pm(M)$ respectively.

Lemma 2.1. *Given a connected orientable smooth 4-manifold M with a fixed orientation, up to unitary transformations the Hilbert spaces $\mathcal{H}^\pm(M)$ are independent of the particular choice of the decomposition used to construct them.*

Proof. Pick two particular decompositions

$$\Omega_c^2(M; \mathbb{C}) = \Omega_c^+(M; \mathbb{C})' \oplus (\Omega_c^-(M; \mathbb{C}))'$$

and

$$\Omega_c^2(M; \mathbb{C}) = \Omega_c^+(M; \mathbb{C})'' \oplus \Omega_c^-(M; \mathbb{C})''$$

into maximal definite pairwise orthogonal subspaces with corresponding Hilbert spaces $\mathcal{H}^\pm(M)'$ and $\mathcal{H}^\pm(M)''$ respectively. Let $U : \Omega_c^2(M; \mathbb{C}) \rightarrow \Omega_c^2(M; \mathbb{C})$ be a unitary transformation with respect to (1) which rotates the first decomposition into the second one. Then obviously the restricted extended maps $U^\pm : \mathcal{H}^\pm(M)' \rightarrow \mathcal{H}^\pm(M)''$ are unitary equivalences hence give rise to Hilbert space isomorphisms $\mathcal{H}^\pm(M)' \cong \mathcal{H}^\pm(M)''$ as claimed. \diamond

Given some decomposition $\Omega_c^2(M; \mathbb{C}) = \Omega_c^+(M; \mathbb{C}) \oplus \Omega_c^-(M; \mathbb{C})$ consider the partially completed space $\mathcal{V}(M) := \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ in the above sense. One can complete $\mathcal{V}(M)$ to a Banach space with respect to the norm

$$\|\omega\| := \left(\|\omega^+\|_{L^2(M)}^2 - \|\omega^-\|_{L^2(M)}^2 \right)^{\frac{1}{2}}$$

where $\mathcal{V}(M) \ni \omega = (\omega^+, \omega^-) \in \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$. Note that this Banach space structure on $\mathcal{V}(M)$ is independent of the particular decomposition $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ we began with.

Summing up, starting with an M we can complete $\Omega_c^2(M; \mathbb{C})$ into a pair $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$ consisting of a complex Banach space $\mathcal{V}(M)$ and a non-degenerate indefinite Hermitian scalar product

$$(\cdot, \cdot)_{L^2(M)} : \mathcal{V}(M) \times \mathcal{V}(M) \longrightarrow \mathbb{C}$$

given by (1) with the following properties. The space admits a family of direct sum Hilbert space structures $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ such that the summands are orthogonal and maximal definite closed subspaces for the scalar product:

$$\begin{cases} \mathcal{H}^+(M) \perp_{L^2(M)} \mathcal{H}^-(M), \\ (\cdot, \cdot)_{L^2(M)}|_{\mathcal{H}^\pm(M)} : \mathcal{H}^\pm(M) \times \mathcal{H}^\pm(M) \longrightarrow \mathbb{C} \text{ are positive or negative definite, respectively.} \end{cases}$$

Definition 2.1. Let M be a connected oriented smooth 4-manifold. The pair $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$ consisting of a Banach space and a non-degenerate indefinite Hermitian scalar product on it is called the generalized Hilbert space of M .

Moreover $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$ carries a representation of $\text{Diff}^+(M)$ from the right given by the unique extension of the pullback of 2-forms: $\omega \mapsto f^* \omega$ for $\omega \in \Omega_c^2(M; \mathbb{C})$ and $f \in \text{Diff}^+(M)$. It is easy to check that this representation is unitary with respect to (1). This representation has the following immediate properties:

Lemma 2.2. Consider the indefinite unitary representation of $\text{Diff}^+(M)$ from the right on the generalized Hilbert space $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$ constructed above.

- (i) (“No vacuum”) A vector $\omega \in \mathcal{V}(M)$ satisfies $f^* \omega = \omega$ for all $f \in \text{Diff}^+(M)$ if and only if $\omega = 0$;
- (ii) The closed subspaces $\mathcal{B}(M) \subseteq \mathcal{Z}(M) \subset \mathcal{V}(M)$ generated by exact or closed 2-forms respectively are invariant under the action of $\text{Diff}^+(M)$.

Proof. (i) Assume that there exists an element $0 \neq \omega \in \mathcal{V}(M)$ stabilized by the whole $\text{Diff}^+(M)$. Consider a 1-parameter subgroup $\{f_t\}_{t \in (-1, +1)} \in \text{Diff}^+(M)$ such that $f_0 = \text{Id}_M$ and let X be the vector field on M generating this subgroup. Differentiating the equation $f_t^* \omega = \omega$ with respect to $t \in (-1, +1)$ at $t = 0$ we obtain $L_X \omega = 0$ where L_X is the Lie derivative by X . Since an arbitrary compactly supported vector field generates a 1-parameter subgroup of $\text{Diff}^+(M)$ we obtain that in fact $\omega = 0$ which is a contradiction.

(ii) Taking into account the naturality of exterior differentiation i.e., $d(f^* \varphi) = f^* d\varphi$ for all $f \in \text{Diff}^+(M)$ and $\varphi \in \Omega_c^k(M; \mathbb{C})$, the statement readily follows. \diamond

Remark. 1. We succeeded to construct a faithful, reducible, unitary representation of the diffeomorphism group out of the structures provided only by an orientable smooth 4-manifold.¹ But the scalar product $(\cdot, \cdot)_{L^2(M)}$ on $\mathcal{V}(M)$ is indefinite therefore there is an extra structure on $\mathcal{V}(M)$ namely a family of decompositions $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ into orthogonal pairs of maximal definite Hilbert

¹In fact our construction so far works in any $4k$ ($k = 1, 2, \dots$) dimensions if the diffeomorphism group acts on $2k$ -forms.

subspaces. Note that such decompositions cannot hold for $\mathcal{V}(M)$ as a $\text{Diff}^+(M)$ -module or in other words such decompositions break the diffeomorphism symmetry. We will see shortly that the classical vacuum Einstein equation says that the vacuum breaks the diffeomorphism invariance in the sense that there is canonical splitting $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ respected by the classical vacuum Einstein equation.

2. We also remark that making use of the structures given by an orientable differentiable manifold only one cannot construct canonically a Hilbert space structure on $\mathcal{V}(M)$ i.e., as a Hilbert space its splittings like $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ are unremovable without using some extra piece of data. In some sense this split Hilbert space structure of the naturally associated vector space is the price one has to pay for maintaining diffeomorphism invariance.

3. From the mathematical viewpoint in many important cases we do not lose any information on the topology of the original manifold if we replace M by the representation space $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$. Indeed, restricting $\Omega_c^2(M; \mathbb{C})$ to closed forms then dividing by the exact ones we can pass to the compactly supported cohomology $H_c^2(M; \mathbb{C})$; then by the Poincaré duality $H_c^2(M; \mathbb{C}) \cong (H^2(M; \mathbb{C}))^*$ we can switch to the ordinary de Rham cohomology. If we assume that M is compact and simply connected then the singular cohomology $H^2(M; \mathbb{Z})$ maps injectively into $H^2(M; \mathbb{C})$ hence finally the scalar product (1) descends to the topological intersection form

$$q_M : H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \longrightarrow H^4(M; \mathbb{Z}) \cong \mathbb{Z}$$

of the underlying topological 4-manifold. However taking into account that by assumption M has a smooth structure we can refer to Freedman's fundamental result [3] that q_M uniquely determines the topology of M .

4. Finally we keep the possibility open that the generalized Hilbert space of our theory to be developed here might be too large (as indicated by part (ii) of Lemma 2.2 for instance) hence might be cut down by some more careful physical or mathematical inspection.

Now we are in a position to construct an *algebraic quantum field theory* on M in the sense of [5] for instance. For this aim we have to associate a *net of local algebras* to M i.e., we need an assignment $U \mapsto \mathcal{A}(U)$ of certain algebras for all relatively compact open $U \subseteq M$ such that the basic axioms of this theory having still meaning in our more general context should be satisfied.

Taking into account that all of our constructions so far are local we can restrict all the objects to open subsets. Fix a relatively compact open subset $U \subseteq M$ and consider the local generalized Hilbert spaces $(\mathcal{V}(U), (\cdot, \cdot)_{L^2(U)})$ carrying an indefinite unitary representation of the local group $\text{Diff}^+(U)$. Given a linear map $A : \mathcal{V}(U) \rightarrow \mathcal{V}(U)$ take its usual operator norm

$$\|A\| := \sup_{0 \neq \omega \in \mathcal{V}(U)} \frac{\|A\omega\|}{\|\omega\|}$$

and write

$$\mathcal{B}(U) := \{A \in \text{End} \mathcal{V}(U) \mid \|A\| < +\infty\}$$

for the C^* -algebra of all bounded linear operators on $\mathcal{V}(U)$. For technical reasons we will rather use its unitization $\widetilde{\mathcal{B}(U)}$ in the usual sense. Since $\mathcal{B}(U)$ has already a unit, we know that as a C^* -algebra we simply have $\widetilde{\mathcal{B}(U)} \cong \mathcal{B}(U) \oplus \mathbb{C}$. Let $\widetilde{\mathcal{B}(U)}$ act on $\mathcal{V}(U)$ from the left by simply putting $(A, z)\omega := A\omega$ for all $(A, z) \in \mathcal{B}(U) \oplus \mathbb{C}$ and $\omega \in \mathcal{V}(U)$. In order to keep our notation simple we will write $\widetilde{\mathcal{B}(U)}$ simply as $\mathcal{B}(U)$ from now on keeping in mind that these local C^* -algebras are always unitized in the sense above.

Our algebraic quantum field theory is then defined by the simple assignment

$$U \longmapsto \mathcal{B}(U), \quad U \subseteq M \text{ is relatively compact open.}$$

Elements of the algebra $\mathcal{B}(U)$ are the *local quantum observables* and those of the group $\text{Diff}^+(U)$ are the *local symmetry transformations*. Consequently observables are given only up to a symmetry transformation i.e., a local orientation preserving diffeomorphism of $U \subseteq M$. The generalized Hilbert space $(\mathcal{V}(U), (\cdot, \cdot)_{L^2(U)})$ carries an action of $\mathcal{B}(U)$ from the left and a unitary one of $\text{Diff}^+(U)$ from the right.² Unit length elements of $\mathcal{V}(U)$ are the *states*. The *expectation value* of $B_U \in \mathcal{B}(U)$ in the state $\omega \in \mathcal{V}(U)$ is defined to be $(\omega, B_U \omega)_{L^2(U)} \in \mathbb{C}$. Note that *a priori* this can be a complex number.

Now we introduce the concept of a quantum gravitational field. The justification of this terminology will be given shortly.

Definition 2.2. *Let M be a connected oriented smooth 4-manifold as above and $U \subseteq M$ a relatively compact open subset.*

- (i) *Then a local observable $Q_U \in \mathcal{B}(U)$ is called a quantum gravitational field localized over U ;*
- (ii) *A quantum gravitational field $Q_U \in \mathcal{B}(U)$ is moreover called a quantum vacuum gravitational field localized over U if it preserves a particular splitting $\mathcal{V}(U) = \mathcal{H}^+(U) \oplus \mathcal{H}^-(U)$ i.e., it satisfies $Q_U(\mathcal{H}^\pm(U)) \subseteq \mathcal{H}^\pm(U)$.*

Remark. 1. Note that we have not required these local observables to be self-adjoint for (1).

2. Elements of the local algebras can be regarded as “operator-valued distributions evaluated on local test functions from a Schwartz-space” i.e., given a hypothetical operator-valued distribution Q over M and $\phi_U \in S(U; \mathbb{C})$ then we suppose that $Q(\phi_U) = Q_U \in \mathcal{B}(U)$ arises by a certain evaluation.

As usual in algebraic quantum field theory quantum gravitational fields behave well under extensions. Take two relatively compact open subsets $U \subseteq V$; then the embedding $e_V^U : U \rightarrow V$ yields an extension by zero $(e_V^U)_* : \Omega_c^2(U; \mathbb{C}) \rightarrow \Omega_c^2(V; \mathbb{C})$. Since by construction our algebras have been unitized this induces a unit-preserving homomorphism $(e_V^U)_* : \mathcal{B}(U) \rightarrow \mathcal{B}(V)$ of unitized C^* -algebras such that the corresponding diagram

$$\begin{array}{ccc} U & \longmapsto & \mathcal{B}(U) \\ e_V^U \downarrow & & \downarrow (e_V^U)_* \\ V & \longmapsto & \mathcal{B}(V) \end{array}$$

is commutative. This allows us to define the algebra $\mathcal{B}(U)$ for any open $U \subseteq M$ or in particular $\mathcal{B}(M)$ for the whole space if M is not compact as the C^* -algebra direct limit of these local algebras. Therefore we can suppose that for all $U \subseteq M$ we simply have $\mathcal{B}(U) \subseteq \mathcal{B}(M)$ and all of these local algebras admit representations on $(\mathcal{V}(M), (\cdot, \cdot)_{L^2(M)})$. However observe that if we consider the dual process namely the restriction $r_U^V : V \rightarrow U$ then quantum gravitational fields do not behave well because the net of algebras does not have the presheaf property in general.

3. Regarding the usual axioms of algebraic quantum field theory (cf. e.g. [5, pp. 105–107]) note that in light of part (i) of Lemma 2.2 it is meaningless to talk about an invariant vacuum in our diffeomorphism-invariant quantum field theory. It is also easy to see that $[\mathcal{B}(U), \mathcal{B}(V)] \neq 0$ if and only if $U \cap V \neq \emptyset$ therefore there is no causality hence no dynamics present here. This is the reason

²Therefore these two actions commute. Hence more inherently we could define the net $\{U \mapsto \mathcal{B}(U)\}_{U \subseteq M}$ by the condition that the induced net of von Neumann algebras $\{U \mapsto N(U)\}_{U \subseteq M}$ should consist of the commutants of the unitary representations of the local symmetry groups. This accords with our original principle that we want to get everything just from seeking a unitary representation of the diffeomorphism group of a 4-manifold.

we will prefer to use Riemannian metrics and not Lorentzian ones in what follows. The only axiom seems to have meaning in this moment is the action of $\text{Diff}^+(M)$ on the net $\{U \mapsto \mathcal{B}(U)\}_{U \subseteq M}$. The naturality of our constructions so far imply that the net of local observables carries a representation of the diffeomorphism group in a natural way:

$$(f^{-1})^*(\mathcal{B}(U))f^* = \mathcal{B}(f(U)) \quad \text{for all } f \in \text{Diff}^+(M).$$

To avoid being short sighted in the investigation of the concept of a quantum gravitational field we have allowed the largest class of linear operators fitting naturally into the algebraic quantum field theory framework; however it is possible that because of some physical or mathematical reasons rather a more restricted class of operators should be introduced. We leave this question open in this moment.

The net of algebras has a natural subset possessing the presheaf property as a consequence of the geometric origin of our construction.

Definition 2.3. Consider the sheaf \mathcal{C}_M over M given by the unitization of local C^* -algebras to be denoted by $\mathcal{C}(U)$ (instead of $\widehat{\mathcal{C}}(U)$, cf. the discussion above) generated by

$$C^\infty(U; \text{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})) \quad \text{for all open } U \subseteq M.$$

(i) A local section $R_U \in \mathcal{C}(U)$ is called a semiclassical gravitational field localized over U ;

(ii) A semiclassical gravitational field $R_U \in \mathcal{C}(U)$ is moreover called a semiclassical vacuum gravitational field localized over U if it preserves a particular splitting $\mathcal{V}(U) = \mathcal{H}^+(U) \oplus \mathcal{H}^-(U)$ i.e., it satisfies $R_U(\mathcal{H}^\pm(U)) \subseteq \mathcal{H}^\pm(U)$.

Remark. 1. Note again that we have not required a semiclassical gravitational field to be self-adjoint with respect to (1).

2. In contrast to general quantum gravitational fields, semiclassical ones behave well under restriction due to their presheaf property. For open subsets $U \subseteq V$ write $r_U^V : V \rightarrow U$ for the restriction. Then the corresponding diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{C}(U) \\ r_U^V \uparrow & & \uparrow (r_U^V)_* \\ V & \longrightarrow & \mathcal{C}(V) \end{array}$$

is commutative.

3. There is an obvious embedding as unital C^* -algebras $\mathcal{C}(U) \subseteq \mathcal{B}(U)$ for all open $U \subseteq M$ consequently these semiclassical fields indeed form a subset of general quantum ones. If $R_U \in \mathcal{B}(U)$ for a relatively compact open $U \subseteq M$ then by definition its norm $\|R_U\| < +\infty$. If in addition $R_U \in \mathcal{C}(U)$ then $\|R_U\| = \|R_U\|_{L^\infty(\overline{U})}$ where

$$\|R_U\|_{L^\infty(\overline{U})} = \sup_{x \in \overline{U}} |R_x|_h$$

and the pointwise length is calculated with respect to some auxiliary Riemannian metric h on M .

Note that from the point of view of the theory of bounded linear operators it would have been more natural to define semiclassical fields as (the unitization of) local C^* -algebras generated by

$$L^\infty(U; \text{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})) \quad \text{for all open } U \subseteq M$$

i.e., to define them by the bounded sections of $\text{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})$ (with respect to an auxiliary Riemannian metric h put onto M). The reason we decided not to do so is that in this case we would loose the sheaf property of the assignment $U \mapsto \mathcal{C}(U)$ which will be used soon (cf. Lemma 2.3 below). However we keep the possibility of a technical modification of this kind as open.

Examples. The time has come to give some examples for various vacuum gravitational fields in the Riemannian setting. These examples also will give an explanation of our terminology.

1. Let (M, g) be a 4-dimensional Riemannian Einstein manifold i.e., assume that g is a Riemannian metric on M with Ricci tensor r_g satisfying the vacuum Einstein equation $r_g = \Lambda_M g$ with a cosmological constant $\Lambda_M \in \mathbb{R}$. In this special situation the vast symmetry group of the original theory cuts down to the tiny one $\text{Iso}^+(M, g) \subsetneq \text{Diff}^+(M)$ consequently observables are given only up to an orientation-preserving isometry of the metric. In this realm the Riemannian metric together with the orientation gives a Hodge operator $*$: $\wedge^2 M \rightarrow \wedge^2 M$ with $*^2 = \text{Id}_{\wedge^2 M}$. This induces a usual splitting

$$\wedge^2 M = \wedge^+ M \oplus \wedge^- M. \quad (2)$$

It is a well-known [9] fact but from our viewpoint is an interesting coincidence that in exactly 4 dimensions the full Riemannian curvature tensor R_g can be regarded as a real linear map $R_g : \wedge^2 M \rightarrow \wedge^2 M$ which decomposes as

$$R_g = \begin{pmatrix} W_g^+ + \frac{s_g}{12} & B_g \\ B_g^* & W_g^- + \frac{s_g}{12} \end{pmatrix}$$

with respect to the splitting (2). Here the traceless symmetric maps $W_g^\pm : \wedge^\pm M \rightarrow \wedge^\pm M$ are the (anti)self-dual parts of the Weyl tensor, the diagonal $s_g : \wedge^2 M \rightarrow \wedge^2 M$ is the scalar curvature while $B_g : \wedge^+ M \rightarrow \wedge^- M$ is the traceless Ricci tensor together with its adjoint $B_g^* : \wedge^- M \rightarrow \wedge^+ M$. Observe that the vacuum Einstein equation $r_g = \Lambda_M g$ exactly says that $B_g = 0$ i.e., $R_g \in C^\infty(M; \text{End}(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C}))$ obeys (2). The pointwise splitting above in addition yields a canonical decomposition

$$\Omega_c^2(M; \mathbb{C}) = \Omega_c^+(M; \mathbb{C}) \oplus \Omega_c^-(M; \mathbb{C})$$

of the space of 2-forms into (anti)self-dual forms which is the same as decomposing this space into mutually orthogonal maximal definite subspaces with respect to the scalar product (1). Putting all of these together and switching to our terminology we conclude that $R_g =: R_M \in \mathcal{C}(M)$ preserves the this time canonical splitting $\mathcal{V}(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M)$ hence is a (real) semiclassical vacuum gravitational field over M . Moreover by the usual symmetries of the curvature tensor R_M is self-adjoint for (1).

Therefore we come up with a natural embedding of classical Riemannian vacuum general relativity into our quantum framework (although note that formally all of our conclusions work for the Lorentzian case, too):

C. *The real Riemannian curvature tensor R_g of an orientable Riemannian Einstein 4-manifold (M, g) can be interpreted as a semiclassical vacuum gravitational field in the sense of part (ii) of Definition 2.3. Moreover the corresponding operator $R_M \in \mathcal{C}(M) \subset \mathcal{B}(M)$ is a real self-adjoint operator with respect to the scalar product (1) on $\mathcal{V}(M)$.*

2. Now consider a connected oriented 4-manifold as usual and pick a generic semiclassical vacuum gravitational field $R_M \in \mathcal{C}(M)$. In this case the previous picture can almost be reversed in the sense that at least locally such observables correspond to Einstein metrics i.e., classical structures. More precisely we claim that

Lemma 2.3. *Let M be a connected oriented smooth 4-manifold and fix an open subset $V \subseteq M$. Let $R_V \in \mathcal{C}(V)$ be a semiclassical vacuum gravitational field. Assume that*

- (i) *In every point $x \in V$ it is real and is an algebraic curvature tensor i.e., the restricted map $R_x : \wedge_x^2 V \rightarrow \wedge_x^2 V$ is real moreover is symmetric and satisfies Bianchi's first identity;*

- (ii) For every point $x \in V$ there exists a non-singular Riemannian metric g'_x on some neighbourhood of x such that its Riemann tensor at x satisfies $R_{g'_x} = R_x$ and its Ricci tensor at x satisfies $r_{g'_x} = \Lambda_x g'_x$ with a constant $\Lambda_x \in \mathbb{R}$.

Then about every point $x \in V$ one can find a neighbourhood $x \in U \subseteq V$ and a local Riemannian metric g_U on U such that

- (i) Its Riemannian curvature coincides with R_V on U more precisely $R_{g_U} = R_V|_U = R_U \in \mathcal{C}(U)$;
(ii) The metric also satisfies the vacuum Einstein equation on U with the local cosmological constant $\Lambda_x \in \mathbb{R}$ i.e., $r_{g_U} = \Lambda_x g_U$ on U .

These local Riemannian Einstein metrics may not fit together into a global one over $V \subseteq M$.

Proof. By referring to a local solvability result of Gasqui [4] for the pseudo-Riemannian curvature the conditions in the lemma guarantee the existence of a neighbourhood $x \in U' \subseteq V$ and a Riemannian metric $g_{U'}$ on U' such that

- (i) $g_{U'} = g'_x$ at least in $x \in U'$;
(ii) $R_{g_{U'}} = R_x$ at least in $x \in U'$;
(iii) $r_{g_{U'}} = \Lambda_x g_{U'}$ along U' with the local cosmological constant $\Lambda_x \in \mathbb{R}$.

We would like to extend part (ii) of this statement from the point x to a whole but probably smaller neighbourhood $x \in U \subseteq U'$. Since by the Definition 2.3 \mathcal{C}_M is a sheaf recall [12] that there exists an isomorphism of sheaves $\tau : \mathcal{C}_M \rightarrow \overline{\mathcal{C}}_M$ where $\overline{\mathcal{C}}_M$ is the étalé space of \mathcal{C}_M defined by $\overline{\mathcal{C}}_M = \bigcup_{x \in M} \mathcal{C}_x$ with \mathcal{C}_x being the stalk of \mathcal{C}_M at $x \in M$. This latter space arises as a direct limit

$$\mathcal{C}_x = \varinjlim_{U \ni x} \mathcal{C}(U)$$

with the well-defined restriction map $r_x^V : \mathcal{C}(V) \rightarrow \mathcal{C}_x$. But note that for any $R_V \in \mathcal{C}(V)$ its value R_x at $x \in V$ satisfies

$$r_x^V(R_V) = R_x$$

consequently $R_x \in \mathcal{C}_x$ i.e., the value of any semiclassical gravitational field at a point can be identified with an element of the corresponding stalk. Then the local Riemannian curvature $R_{g_{U'}}$ obviously satisfies $R_{g_{U'}} \in \mathcal{C}(U')$ and part (ii) of Gasqui's result shows that in addition

$$r_x^{U'}(R_{g_{U'}}) = R_x.$$

However these last two equations imply by the aid of the definition of a stalk that there exists a neighbourhood $x \in U \subseteq U'$ such that in fact

$$r_U^V(R_V) = r_U^{U'}(R_{g_{U'}}).$$

Therefore we put the neighbourhood around x to be this U and the metric on it to be the restriction $g_U := r_U^{U'}(g_{U'})$.

Moreover part (iii) of Gasqui's theorem obviously provides us that g_U also satisfies the local Einstein equation with cosmological constant $\Lambda_x \in \mathbb{R}$ as desired. \diamond

We conclude that a generic semiclassical vacuum gravitational field can be interpreted as a certain structure on the manifold which at least locally resembles to classical Riemannian general relativity (we note again that all of these conclusions are valid in the Lorentzian case, too):

SC. *A semiclassical vacuum gravitational field $R_M \in \mathcal{C}(M) \subset \mathcal{B}(M)$ in the sense of Definition 2.3 (and satisfying some technical conditions as stated in Lemma 2.3) over a connected oriented smooth 4-manifold M give rise to local Riemannian Einstein metrics around the points of M .*

3. Finally we take a further departure from classical general relativity and explore the deep quantum regime as defined in part (ii) of Definition 2.2. Of course the trouble is how to describe a quantum field i.e., a generic bounded linear operator $Q_M \in \mathcal{B}(M)$ in terms of a semiclassical one i.e., a geometric linear operator $R_M \in \mathcal{C}(M) \subsetneq \mathcal{B}(M)$. Our quantum instinct tells us that a truly quantum operator should be constructed by somehow smearing a semiclassical operator over regions in M .

Fix a semiclassical operator $R_U \in \mathcal{C}(U)$ and a point $x \in U$. Then on any 2-form $\omega \in \Omega_c^2(U; \mathbb{C})$ its action can be expressed in a fully local form $(R_U \omega)_x = R_x \omega_x$. Motivated by the Ambrose–Singer holonomy theorem [1] we can generalize this as follows. Pick finitely many distinct further points $y_1, \dots, y_{n(x)} \in U$ where $n(x) \in \mathbb{N}$ may depend on $x \in U$. Consider diffeomorphisms $f_i \in \text{Diff}^+(U)$ such that $f_0 = \text{Id}_U$ hence $f_0(x) = x$ moreover $f_i(x) = y_i$ for $i = 1, \dots, n(x)$. Let us construct an operator $Q_U \in \mathcal{B}(U)$ out of $R_U \in \mathcal{C}(U)$ and $f_0, \dots, f_{n(x)} \in \text{Diff}^+(U)$ such that on vectors $\omega \in \Omega_c^2(U; \mathbb{C}) \subset \mathcal{V}(U)$ forming a dense subset in any point $x \in U$ it has the shape

$$(Q_U \omega)_x := \sum_{i=0}^{n(x)} f_i^*(R_U \omega) = R_x \omega_x + \sum_{i=1}^{n(x)} f_i^*(R_U \omega). \quad (3)$$

Note that this linear operator is bounded but not local in the sense that its effect on ω_x depends not only on R_x and ω_x but on the value of R_U and ω in further distant points $y_1, \dots, y_{n(x)} \in U$ as well.

The question arises how to generalize this construction for infinite sums or even integrals. Recall that a *double 2-form* K is a section of the bundle $(\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C}) \times (\wedge^2 M \otimes_{\mathbb{R}} \mathbb{C})$ over $M \times M$ which is in general supposed to be smooth over $M \times M \setminus D$ where $D := \{(x, y) \in M \times M \mid x = y\}$ is the diagonal and possesses some singular behaviour along $D \subset M \times M$. Now we are in a position to generalize the discrete expression (3) as follows.

Lemma 2.4. *Let M be a connected, orientable smooth 4-manifold, $U \subseteq M$ an open subset. Then for any bounded linear operator $Q_U \in \mathcal{B}(U)$ i.e., a quantum gravitational field there exists at least one geometric operator $R_U \in \mathcal{C}(U)$ i.e., a semiclassical gravitational field and a double 2-form K such that on 2-forms $\omega \in \Omega_c^2(U; \mathbb{C})$ forming a dense subset in $\mathcal{V}(U)$ the operator looks like*

$$(Q_U \omega)_x = \int_{y \in U} K_{x,y} \wedge (R_U \omega)_y.$$

This integral always exists because $\omega \in \Omega_c^2(U; \mathbb{C})$ has compact support.

Note that if $R_U \in \mathcal{C}(U)$ is a semiclassical vacuum quantum gravitational field in the sense of part (ii) of Definition 2.3 then $Q_U \in \mathcal{B}(U)$ constructed out of R_U and any K this way is also vacuum in the sense of part (ii) of Definition 2.2. \diamond

Proof. This follows from the Schwartz kernel theorem cf. e.g. [10, Vol. I. Section 4.6]. \diamond

Remark. We can link this representation of Q_U to the discrete version (3) more directly as follows. For all points $y \in U$ pick up unique diffeomorphisms $f_y \in \text{Diff}^+(U)$ such that $f_y(x) = y$ and $f_x = \text{Id}_U$. Then for all $\omega \in \Omega_c^2(U; \mathbb{C})$ the assignment $y \mapsto f_y^*(R_U \omega)$ gives a function from U into $\wedge_x^2 U \otimes_{\mathbb{R}} \mathbb{C}$. Suppose moreover that there is a complex *measure* μ_x on U . Then the kernel K is a right tool how to integrate this function against this measure i.e., for all $x \in U$ and $\omega \in \Omega_c^2(U; \mathbb{C})$ we put

$$\int_{y \in U} f_y^*(R_U \omega) d\mu_x(y) := \int_{y \in U} K_{x,y} \wedge (R_U \omega)_y \in \wedge_x^2 U \otimes_{\mathbb{R}} \mathbb{C}.$$

The difficulty is that without additional structures one cannot find a measure on M for integration.

We summarize again our findings as follows:

Q. A general quantum gravitational (vacuum) field $Q_M \in \mathcal{B}(M)$ in the sense of Definition 2.2 over a connected oriented smooth 4-manifold M can be constructed from a semiclassical (vacuum) one $R_M \in \mathcal{C}(M)$ in the sense of Definition 2.3 by a smearing procedure formulated in Lemma 2.4. In this general situation no pointwisely given geometric object has a meaning in accord with the physical expectations.

We have completed the exploration of quantum gravitational fields in our model.

3 Some calculations

Reversing the construction outlined in Lemma 2.4 we can “quantize” a classical vacuum gravitational field i.e., a Riemannian Einstein 4-manifold (M, g) with $r_g = \Lambda_M g$ as follows. Since the symmetry group $\text{Diff}^+(M)$ is cut down to $\text{Iso}^+(M, g)$ in the classical case we can talk about the Hodge operator $*$ and the Laplacian Δ^2 on 2-forms. Let G denote a “Green function” for Δ^2 regarded as a double 2-form over $(M \times M, g \times g)$. Such a Green function is given up to an element of $H^2(M; \mathbb{C})$. Then a natural choice for the kernel is $K_{x,y} := *_y G_{x,y}$. Viewing the full Riemannian curvature R_g as a semiclassical vacuum gravitational field $R_M \in \mathcal{C}(M)$ we can construct a quantum gravitational field $Q_M \in \mathcal{B}(M)$ by

$$(Q_M \omega)_x := \int_{y \in M} G_{x,y} \wedge *_y (R_M \omega)_y$$

on densely defined elements $\omega \in \Omega_c^2(M; \mathbb{C}) \subset \mathcal{V}(M)$.

For example the case of the flat space (\mathbb{R}^4, g) is very simple. This geometry has identically vanishing curvature $R_{\mathbb{R}^4} = 0$. Consequently for all $x \in \mathbb{R}^4$

$$(Q_{\mathbb{R}^4} \omega)_x = \int_{y \in \mathbb{R}^4} G_{x,y} \wedge *_y (0 \cdot \omega)_y = 0 \in \wedge_x^2 \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}.$$

Therefore in this case $Q_{\mathbb{R}^4} \in \mathcal{B}(\mathbb{R}^4)$ i.e., $Q_{\mathbb{R}^4} : \mathcal{V}(\mathbb{R}^4) \rightarrow \mathcal{V}(\mathbb{R}^4)$ simply looks like $Q_{\mathbb{R}^4} = 0$. We get the same trivial result for the 27 orientable compact flat 4-manifolds (in total there are as many as 74 compact flat manifolds in four dimensions), cf. [7].

Another example is the round sphere (S^4, g) considered in the flat \mathbb{R}^5 with radius $r \in \mathbb{R}^+$. The only non-trivial component of R_g is its scalar part: $R_y = \frac{s_g(y)}{12} \text{Id}_{\wedge_y^2 S^4 \otimes_{\mathbb{R}} \mathbb{C}}$ with $s_g(y) = \frac{12}{r^2}$ constant. Hence for all $x \in S^4$ we obtain

$$(Q_{S^4} \omega)_x = \frac{1}{r^2} \int_{y \in S^4} G_{x,y} \wedge *_y \omega_y \in \wedge_x^2 S^4 \otimes_{\mathbb{R}} \mathbb{C}.$$

In this very simple but already not trivial situation it is easy to find the corresponding eigenvectors $Q_{S^4}\omega = \lambda\omega$ with some $\lambda \in \mathbb{C}$. In fact Q_{S^4} is a compact self-adjoint operator hence its eigenvectors exist, span $\mathcal{V}(S^4)$ and the eigenvalues form a bounded subset of $\mathbb{R} \subset \mathbb{C}$ with the only accumulation point $0 \in \mathbb{R}$. Taking into account that $H^2(S^4; \mathbb{C}) = 0$ the kernel of Δ^2 is trivial hence its Green function G is unique moreover $\Delta_x^2 G_{x,y} = \text{Id}_{\wedge_x^2 S^4 \otimes_{\mathbb{R}} \mathbb{C}} \delta_x$ in the distributional sense. Therefore applying Δ^2 to $Q_{S^4}\omega = \lambda\omega$ i.e., to

$$\frac{1}{r^2} \int_{y \in S^4} G_{x,y} \wedge *_y \omega_y = \lambda \omega_x$$

we obtain $\frac{1}{r^2}\omega = \lambda\Delta^2\omega$. In summary we conclude that if $\omega \in \mathcal{V}(S^4)$ is an eigenvector of Q_{S^4} with eigenvalue λ then it is an eigenvector of Δ^2 with eigenvalue $\frac{1}{r^2\lambda}$. Moreover in fact $\omega \in \Omega^2(S^4; \mathbb{C})$.

Finally we consider the Riemannian Schwarzschild space (M, g) . In contrast to the Lorentzian original, this Wick rotated version is an everywhere smooth complete Ricci flat space, cf. [11, p. 407]. In this case $M = S^2 \times \mathbb{R}^2$ with the usual spherical coordinates (Θ, ϕ) on S^2 and polar coordinates (r, τ) on \mathbb{R}^2 . The metric g on a dense open subset of $S^2 \times \mathbb{R}^2$ given by

$$0 \leq \tau < 8\pi m, \quad 2m \leq r < +\infty, \quad 0 \leq \Theta < \pi, \quad 0 \leq \phi < 2\pi$$

has the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\Theta^2 + \sin^2 \Theta d\phi^2).$$

In any point $x = (\Theta, \phi, r, \tau) \in S^2 \times \mathbb{R}^2$ of the coordinate range we choose an orthonormal frame for the metric

$$\xi^\tau := \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} d\tau, \quad \xi^r := \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr, \quad \xi^\Theta := rd\Theta, \quad \xi^\phi := r \sin \Theta d\phi$$

to span $\wedge_x^2(S^2 \times \mathbb{R}^2)$ with $\xi^\Theta \wedge \xi^\phi$, etc. The curvature operator $R_{\text{Schw},x} : \wedge_x^2(S^2 \times \mathbb{R}^2) \rightarrow \wedge_x^2(S^2 \times \mathbb{R}^2)$ in this basis looks like

$$R_{\text{Schw},(\Theta,\phi,r,\tau)} = \frac{2m}{r^3} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(only the two Weyl tensors are not zero and are equal). Let T be the endomorphism of $\wedge^2(S^2 \times \mathbb{R}^2)$ with $T_y = \text{Diag}(2, -1, -1, 2, -1, -1)$ in the orthonormal frame over every y . With these notations $(R_{\text{Schw}}\omega)_y = \frac{2m}{r^3(y)}(T\omega)_y$. Green functions are not unique because $H^2(S^2 \times \mathbb{R}^2; \mathbb{C}) \cong \mathbb{C}$. Taking any of them we come up with

$$(Q_{\text{Schw}}\omega)_x = \int_{y \in S^2 \times \mathbb{R}^2} \frac{2m}{r^3(y)} G_{x,y} \wedge *_y (T\omega)_y \in \wedge_x^2(S^2 \times \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$$

for all $x \in S^2 \times \mathbb{R}^2$ and $\omega \in \Omega_c^2(S^2 \times \mathbb{R}^2; \mathbb{C}) \cap \mathcal{V}(S^2 \times \mathbb{R}^2)$. Note again that this integral is convergent because ω has compact support on $S^2 \times \mathbb{R}^2$.

4 Conclusion and outlook

From the genuine viewpoint of algebraic quantum field theory we constructed out of an oriented smooth 4-manifold M an abstract C^* -algebra $\mathcal{B}(M)$ such that this algebra contains (i) algebraic curvature tensors on M and (ii) as its automorphisms the group $\text{Diff}^+(M)$. In this way classical general relativity naturally embeds into a quantum framework. Moreover this abstract C^* -algebra admits a representation on a Banach space again constructed only from the manifold and its orientation. This space in addition has split Hilbert space structures and quantum observables for Einstein manifolds obey a canonical one of them yielding a superselection rule interpretation of the vacuum Einstein equation. It is interesting that all of these work only in four dimensions.

The straightforward generalization is to couple gravity with matter and try to work out an analogous inclusion. This would be especially important since in our understanding—as we argued in the Introduction—the presence of matter is necessary to recover not only the Riemannian geometric but the Lorentzian causal structure of the space-time, too. Quantum gravitational fields provided by gravity coupled to matter are expected to break the superselection rule valid in the pure gravity case and they are generally not self-adjoint.

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